

Introduction

Limit Theorems

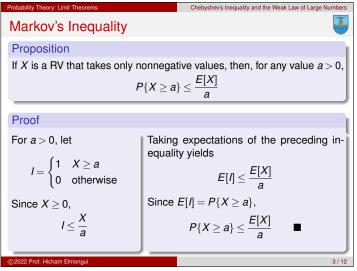
- The most important theoretical results in probability theory.
- The most important theorems are those classified either under *t Laws of large numbers*
 - ★ Central limit theorems

Laws of large numbers

They are concerned with stating conditions under which the average of a sequence of RVs converges to the expected average.

Central limit theorems

They are concerned with determining conditions under which the sum of a large number of RVs has a probability distribution that is approximately normal.



Probability Theory: Limit Theorems	Chebyshev's Inequality and the Weak Law of Large Number
Chebyshev's Inequality	
Proposition	
If X is a RV with finite mean μ and v	ariance σ^2 , then, for any value $k>0$
$P\{ X-\mu $	$\geq k \} \leq \frac{\sigma^2}{k^2}$
Proof	
Since $(X - \mu)^2$ is a nonnegative random variable, we can apply Markov's inequality (with $a = k^2$) to obtain	
$P\left\{(X-\mu)^2 \ge k^2 ight\} \le rac{E[(X-\mu)^2]}{k^2}$	
But since $(X - \mu)^2 \ge k^2$ if and only if $ X - \mu \ge k$,	
$P\{ X-\mu \geq k\}\leq \frac{E }{2}$	$\frac{[(X-\mu)^2]}{k^2} = \frac{\sigma^2}{k^2} \qquad \blacksquare$
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Examples

Chebyshev's Inequality and the Weak Law of Large Numbers

Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.

- What's the probability that this week's production will exceed 75?
- If the variance of a week's production is found to be 25, what's the probability that this week's production will be between 40 and 60?

Solution

Let *X* be the number of items that will be produced in a week.

• By Markov's inequality,

$$P\{X > 75\} \le \frac{E[X]}{75} = \frac{50}{75} = \frac{2}{3}$$

• By Chebyshev's inequality,
 $P\{|X - 50| \ge 10\} \le \frac{\sigma^2}{10^2} = \frac{25}{10^2} = \frac{1}{4}$
 $P\{|X - 50| < 10\} \ge 1 - \frac{1}{4} = \frac{3}{4}$

Examples (cont'd)

As Chebyshev's inequality is valid for all distributions of the random variable X, we cannot expect the bound on the probability to be very close to the actual probability in most cases.

T

6/12

Example

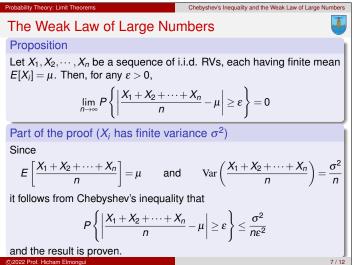
If
$$X \sim \text{Normal}(\mu, \sigma^2)$$
, Chebyshev's inequality states that

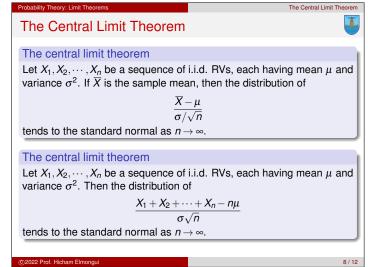
 $P\{|X-\mu|>2\sigma\}\leq \frac{1}{4}$

whereas the actual probability is given by

$$P\{|X - \mu| > 2\sigma\} = P\left\{ \left| \frac{X - \mu}{\sigma} \right| > 2 \right\} = 2\left(1 - \Phi(2)\right) \approx .0456$$

Chebyshev's inequality is often used as a theoretical tool in proving results.





Probability Theory: Limit Theorems	The Central Limit Theorem	
Examples	1	
If 10 fair dice are rolled, find the approximate probability that the sum obtained is between 30 and 40, inclusive.		
Solution		
Let X_i denote the value of the i^{th} die	e, $i = 1, 2, \cdots, 10$. Since	
$E[X_i] = \frac{7}{2}$	$\operatorname{Var}(X_i) = \frac{35}{12}$	
the central limit theorem yields		
$P\{29.5 \le X \le 40.5\} = P\left\{\frac{29.5}{V}\right\}$	$\frac{5-35}{\frac{350}{12}} \le \frac{X-35}{\sqrt{\frac{350}{12}}} \le \frac{40.5-35}{\sqrt{\frac{350}{12}}} \right\}$	
$pprox P\{-1.0$	$184 \le Z \le 1.0184$	
$=\Phi(1.018$	34) – Φ(–1.0184)	
= 2Φ(1.01	84) − 1 ≈ .692	
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Examples (cont'd) An instructor has 50 exams that will be graded in sequence. The times required to grade the 50 exams are independent, with a common distribution that has mean 20 minutes and standard deviation 4 minutes. Approximate the probability that the instructor will grade at least 25 of the exams in the first 450 minutes of work. Solution If we let X_i be the time that it takes to grade exam *i*, then the time it takes to grade the first 25 exams is $X = \sum_{i=1}^{25} X_i$ $E[X] = \sum_{i=1}^{25} E[X_i] = 25(20) = 500$ $P\{X \le 450\}$ (X - 500 - 450 - 500)

The Central Limit Theorem

$$E[X] = \sum_{i=1}^{2} E[X_i] = 25(20) = 500$$

$$Var(X) = \sum_{i=1}^{25} Var(X_i) = 25(16) = 400$$

$$= P\left\{\frac{X - 500}{\sqrt{400}} \le \frac{450 - 500}{\sqrt{400}}\right\}$$

$$\approx P\{Z \le -2.5\}$$

$$= 1 - \Phi(2.5) \approx .006$$

10/12

